GENERALIZED LADDER OPERATORS FOR THE DIRAC-COULOMB PROBLEM VIA SUSY QM

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Abstract

Supersymmetry and the shape invariance condition in quantum mechanics are applied as an algebraic method to solve the Dirac-Coulomb problem. The ground state and the excited states are investigated using new generalized ladder operators.

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I. INTRODUCTION

The supersymmetry (SUSY) algebra in quantum mechanics (QM) began with the work of Nicolai [1] and was elegantly formulated by Witten [2]. This approach has attracted interest and was applied to construct the spectral resolution of solvable potentials in many areas of physics [3]. SUSY QM was also formulated by Gendenshtein who used the shape invariance property [4]. The hydrogen atom was studied via SUSY QM in the non-relativistic context by Kostelecky and Nieto [5]. They used the SUSY QM for spectral resolution and also for calculating transition probabilities for alkali-metal atoms. Also, Zhang et al. have considered interesting applications of a semi-unitary formulation of SUSY QM [6].

The Dirac-Coulomb problem is an exactly solvable problem in relativistic quantum mechanics and the solution can be found in books on quantum mechanics, see for instance [7]. The Dirac-Coulomb problem has also been studied via SUSY QM [9–17]. Our purpose in this paper is to obtain the complete energy spectrum and the energy eigenfunctions of the Dirac-Coulomb problem using the shape invariance and new generalized ladder operators. The Lie algebra associated with these operators for the shape invariant potentials has been presented by Fukui-Aizawa [18] and Balantekin [19]. This formalism has been applied to the exactly solvable potentials in non-relativistic quantum mechanics [20,21].

It is particularly simple to apply SUSY QM to shape-invariant potentials because their SUSY partners are similar in shape and differ only in the parameters that appear in them. More specifically, if $V_-(x; a_1)$ is any potential, adjusted to have zero ground state energy $E_-^{(0)} = 0$, its SUSY partner $V_+(x; a_0)$ must satisfy the requirement $V_+(x; a_0) = V_-(x; a_1) + R(a_1)$, where a_0 is a set of parameters, a_1 a function of the parameter a_0 and $R(a_1)$ is independent of x. In this case, one can determine the energy levels for $V_+(x; a_0)$, which correspond to $E_-^{(n)} = \sum_{i=1}^{n} R(a_i)$. In this context, recently, some relativistic shape invariant potentials have been investigated [22].

The SUSY hierarchical prescription was used by Sukumar to solve the energy spectrum of the Dirac-Coulomb problem [10]. In this work the Fukui-Aizawa-Balantekin [18,19] approach to the Dirac-Coulomb problem is investigated using SUSY QM.

The paper is organized as follows. In section II we realize a graded Lie algebra structure in terms of the 4x4 matrix supercharges, analogous to Witten's SUSY algebra for the Dirac radial equation associated with the hydrogen atom. In section III, using the shape invariance condition we deduce new generalized ladder operators in relativistic quantum mechanics, via supersymmetry, in order to build up the energy eigenvalue and eigenfunctions of supersymmetric partner potentials. In section IV some concluding remarks are given.

II. THE DIRAC-COULOMB PROBLEM AND SUSY

In this section, we adopt the Sukumar's approach to construct the 4x4 matrix supercharges. The Dirac radial equation for the hydrogen atom can be written as

$$\begin{pmatrix} \frac{dG}{dr} & 0 \\ 0 & \frac{dF}{dr} \end{pmatrix} + \frac{1}{r} \begin{pmatrix} k & -\gamma \\ \gamma & -k \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}, \tag{1}$$

where k is an eigenvalue of the Dirac operator $K = \beta(\vec{\Sigma} \cdot \vec{L} + \mathbf{1})$, $\gamma = \frac{ze^2}{c\hbar}$, $\alpha_1 = m + E$, $\alpha_2 = m - E$, $|k| = j + \frac{1}{2}$ $(k = \pm 1, \pm 2, \pm 3, \cdots), \vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$ are a set of the

Pauli matrices, $\vec{L}$ is the angular momentum operator and $\mathbf{1}$ is the 2x2 unity matrix. Now, let D be the operator which diagonalizes the matrix that appears in the interaction term

$$D^{-1}(k\sigma_3 - i\gamma\sigma_2)D = s\sigma_3, \tag{2}$$

where $s = (k^2 - \gamma^2)^{\frac{1}{2}}$. Then we get

$$D = (s + k)\mathbf{1} + \gamma\sigma_1. \tag{3}$$

Thus, we obtain

$$\left(\frac{k}{s} + \frac{m}{E}\right)\tilde{F} = \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right)\tilde{G}, \tag{4a}$$

$$\left(\frac{k}{s} - \frac{m}{E}\right)\tilde{G} = \left(-\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right)\tilde{F}, \tag{4b}$$

where

$$\begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} = D \begin{pmatrix} G \\ F \end{pmatrix}, \quad \rho = Er. \tag{5}$$

The eigenvalue equations for $k = |k|$ and $k = -|k|$, respectively, become

$$\left(\frac{|k|}{s} + \frac{m}{E}\right)\tilde{F}_+ = \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right)\tilde{G}_+, \tag{6a}$$

$$\left(\frac{|k|}{s} - \frac{m}{E}\right)\tilde{G}_+ = \left(-\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right)\tilde{F}_+, \tag{6b}$$

$$\left(\frac{-|k|}{s} + \frac{m}{E}\right)\tilde{F}_- = \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right)\tilde{G}_-, \tag{6c}$$

$$\left(\frac{-|k|}{s} - \frac{m}{E}\right)\tilde{G}_- = \left(-\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right)\tilde{F}_-, \tag{6d}$$

where $\tilde{F}_\pm = \tilde{F}(\pm|k|)$ and $\tilde{G}_\pm = \tilde{G}(\pm|k|)$.

Defining the intertwining operators as

$$A_0^{(+)} = \mathbf{I}\frac{d}{d\rho} + \left(\frac{s}{\rho} - \frac{\gamma}{s}\right)\sigma_3, \tag{7}$$

$$A_0^{(-)} = \left(A_0^{(+)}\right)^\dagger = -\mathbf{I}\frac{d}{d\rho} + \left(\frac{s}{\rho} - \frac{\gamma}{s}\right)\sigma_3 \tag{8}$$

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where $\mathbf{I}$ denotes the 2x2 unit matrix and

$$\mathbf{O} = \frac{m}{E}\sigma_1 + i\frac{\mid k \mid}{s}\sigma_2, \tag{9}$$

we get

$$A_0^{(+)}\begin{pmatrix}\tilde{G}_+\\\tilde{F}_+\end{pmatrix} = \mathbf{O}\begin{pmatrix}\tilde{G}_+\\\tilde{F}_+\end{pmatrix}, \tag{10}$$

$$A_0^{(-)}\begin{pmatrix}\tilde{F}_-\\\tilde{G}_-\end{pmatrix} = -\mathbf{O}\begin{pmatrix}\tilde{F}_-\\\tilde{G}_-\end{pmatrix}, \tag{11}$$

where $\mathbf{I}$ denotes the 2x2 unit matrix.

Using the result

$$A_0^- = -\sigma_1 A_0^+ \sigma_1, \tag{12}$$

we see that there exist the supersymmetric partner eigenvalue equations:

$$A_0^{(-)}A_0^{(+)}\begin{pmatrix}\tilde{G}_+\\\tilde{F}_+\end{pmatrix} = \left(1 + \frac{\gamma^2}{s^2} - \frac{m^2}{E^2}\right)\begin{pmatrix}\tilde{G}_+\\\tilde{F}_+\end{pmatrix} \tag{13}$$

and

$$A_0^{(+)}A_0^{(-)}\begin{pmatrix}\tilde{F}_-\\\tilde{G}_-\end{pmatrix} = \left(1 + \frac{\gamma^2}{s^2} - \frac{m^2}{E^2}\right)\begin{pmatrix}\tilde{F}_-\\\tilde{G}_-\end{pmatrix}. \tag{14}$$

The mutually adjoint non-Hermitian supercharge operators for Witten's model are given by

$$Q_+ = A_0^{(+)}\sigma_- = \begin{pmatrix}0 & A_0^{(+)}\\0 & 0\end{pmatrix}_{4x4}, \quad Q_- = A_0^{(-)}\sigma_+ = \begin{pmatrix}0 & 0\\A_0^{(-)} & 0\end{pmatrix}_{4x4}, \tag{15}$$

so that the SUSY Hamiltonian H takes the form

$$H = [Q_+, Q_-]_+ = \begin{pmatrix}H_- & 0\\0 & H_+\end{pmatrix}, \tag{16}$$

and satisfies $[H, Q_\pm]_- = 0$.

The pair of SUSY hamiltonians is given by

$$H_\pm = -\mathbf{I}\frac{d^2}{d\rho^2} + W^2 \mp \frac{dW}{d\rho} \tag{17}$$

where the matrix superpotential is given by $W(\rho) = \left(\frac{s}{\rho} - \frac{\gamma}{s}\right)\sigma_3$.

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III. SPECTRAL RESOLUTION VIA LADDER OPERATORS

Let us now build up the energy eigenvalues and eigenfunctions of supersymmetric partner potentials using the shape invariance condition and the generalized ladder operators. From the last section one obtains the following matrix forms of the pair of SUSY potentials:

$$V_-(\rho, \lambda, s) = \begin{pmatrix} \frac{s(s-1)}{\rho^2} - \frac{2\gamma}{\rho} + \frac{\gamma^2}{s^2} & 0 \\ 0 & \frac{s(s+1)}{\rho^2} - \frac{2\gamma}{\rho} + \frac{\gamma^2}{s^2} \end{pmatrix}, \tag{18}$$

$$V_+(\rho, \lambda, s) = \begin{pmatrix} \frac{s(s+1)}{\rho^2} - \frac{2\gamma}{\rho} + \frac{\gamma^2}{s^2} & 0 \\ 0 & \frac{s(s-1)}{\rho^2} - \frac{2\gamma}{\rho} + \frac{\gamma^2}{s^2} \end{pmatrix}. \tag{19}$$

Although the SUSY partner potentials $V_{(\pm)}$ are not shape invariant, we can see that their components are:

$$V_{(+)11}(\rho, \gamma, s) = V_{(-)11}(\rho, \gamma, s+1) - \frac{\gamma^2}{(s+1)^2} + \frac{\gamma^2}{s^2}, \tag{20}$$

$$V_{(-)22}(\rho, \gamma, s) = V_{(+)22}(\rho, \gamma, s+1) - \frac{\gamma^2}{(s+1)^2} + \frac{\gamma^2}{s^2}. \tag{21}$$

From (26), (20) and (21) one can written

$$R_{11}(a_1) = R_{22}(a_1) = -\frac{\gamma^2}{(s+1)^2} + \frac{\gamma^2}{s^2} = \frac{\gamma^2}{a_0^2} - \frac{\gamma^2}{a_1^2}, \tag{22}$$

where $a_0 = s$ and $a_1 = a_0 + 1$, so that $a_i = a_0 + i$, $R_{11}(a_i) = \frac{-\gamma^2}{(s+i)^2} + \frac{\gamma^2}{s^2} = \frac{-\gamma^2}{(a_i)^2} + \frac{\gamma^2}{a_0^2}$. Thus we get the following energy eigenvalues of $H_{(-)11} = H_{(+)22}$:

$$E_{-11}^{(n)} = E_{+22}^{(n)} = \sum_{i=1}^{n} R_{11}(a_i) = -\gamma^2 \sum_{i=1}^{n} \left(\frac{1}{(a_i)^2} - \frac{1}{a_{i-1}^2} \right) = \frac{-\gamma^2}{(s+n)^2} + \frac{\gamma^2}{s^2}. \tag{23}$$

Comparing (23) with the eigenvalue equation (14) we obtain the energy eigenvalues of the hydrogen relativistic atom, viz.,

$$1 + \frac{\gamma^2}{s^2} - \frac{m^2}{E^{(n)2}} = \frac{\gamma^2}{s^2} - \frac{\gamma^2}{(s+n)^2}, \tag{24}$$

providing

$$E^{(n)} = \sqrt{\frac{m^2}{1 + \frac{\gamma^2}{(\sqrt{k^2 - \gamma^2} + n)^2}}}, \quad n = 0, 1, 2, \cdots, \tag{25}$$

which is in agreement with the result obtained by Sukumar using the SUSY Hamiltonian hierarchy method [10].

Note that the shape invariance condition is associated with translation of the parameters a's, so that the Eq. (20) can be written in the following form

$$A_{11}^{(-)}(a_0)A_{11}^{(+)}(a_0) = A_{11}^{(+)}(a_1)A_{11}^{(-)}(a_1) + R_{11}(a_1), \tag{26}$$

where

$$A_{11}^{(\pm)}(s) = \pm\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}, \quad a_0 = s. \tag{27}$$

Following Fukui-Aizawa-Balantekin approach [18,19], we obtain the following ladder operators

$$B_-(s) = T^\dagger(s)A_{11}^-(s), \quad B_+(s) = B_-^\dagger(s), \tag{28}$$

where $T(s)$ being a translation operator defined by

$$T(s) = e^{\frac{\partial}{\partial s}}, \tag{29}$$

with $T^\dagger(s) = e^{-\frac{\partial}{\partial s}}$. In this case we have the identity $R(a_n) = T(a_0)R(a_{n-1})T^\dagger(a_0)$ and the folowing algebra

$$[B-, B_+] = T(a_0)R(a_0)T^\dagger(a_0) = R(a_1), \quad R(a_n)B_+(a_0) = B_+(a_0)R(a_{n-1}), \tag{30}$$

where the shape invariance provides us the translations of the parameters a_n, viz., $a_n = a_0 + n$, valid for any n. Thus, it is easy to see that the operators $B_\pm(a_0)$ and $R(a_n)$ satisfy the following commutation relations

$$[H_{(-)11}, B_+^n] = (R(a_1) + R(a_2) + \cdots + R(a_n))B_+^n, \tag{31}$$

and

$$[H_{(-)11}, B_-^n] = -B_-^n(R(a_1) + R(a_2) + \cdots + R(a_n)), \quad n = 1, 2, \cdots, \tag{32}$$

with $H_{(-)11} = B_+B_- = A_{11}^{(+)}A_{11}^{(-)}$. Consequently we see that the $\tilde{F}_-^{(n)}$, component eigenfunction of the n-th excited stated, is given by

$$\tilde{F}_-^{(n)} \propto B_+^n(s)\tilde{G}_-^{(0)}(\rho; s), \quad n = 1, 2, 3, \cdots. \tag{33}$$

The ground state eigenfunction must be annihilated by $B_-(s)$, then

$$A^-(s)\tilde{G}_-^{(0)}(\rho; s) = 0, \tag{34}$$

which leads us to the following physically acceptable solution

$$\tilde{G}_-^{(0)}(\rho; s) = N_G \rho^s e^{-\frac{\gamma}{s}\rho}, \tag{35}$$

where N_G being the normalization constant.

¿From (28) and (29) we see that the raising operator may be written as

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$$B_+(s) = \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s} \right) e^{\frac{\partial}{\partial s}}. \tag{36}$$

Consequently, for the first excited state one may write

$$\begin{aligned}
\tilde{F}_-^{(1)}(\rho; s) &= B_+(s)\tilde{G}_-^{(0)}(\rho; s) \\
&\propto \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s} \right) e^{\frac{\partial}{\partial s}} \rho^s e^{-\frac{\gamma}{s}\rho} \\
&= \left[(2s+1) \left(\frac{1}{\rho} - \frac{\gamma}{s(s+1)} \right) \right] \rho^{s+1} e^{-\frac{\gamma\rho}{s+1}}.
\end{aligned} \tag{37}$$

Finally we would like to call attention to the fact that the above formalism may be applied to the exactly solvable potentials in relativistic quantum mechanics [22,23]. Also, the applications of semi-unitary transformations to construct supersymmetric partner Hamiltonian in non-relativistic quantum mechanics [6] can be implemented for the Dirac-Coulomb problem.

IV. CONCLUSION

In this paper we investigated the Dirac-Coulomb problem via supersymmetry in quantum mechanics. The shape invariant formalism for the supersymmetric partners is applied to obtain the complete energy spectrum and eigenfunctions of the Dirac-Coulomb problem.

Our approach uses the algebraic structure for shape invariant potential, recently proposed by Fukui-Aizawa-Balantekin [18,19]. This approach is different from the SUSY Hamiltonian hierarchy method applied by Sukumar [10].

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